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Technical Comments

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Comment on "Exact Dynamic Analysis of Space Structures Using Timoshenko Beam Theory"

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FREE vibration analysis of three-dimensional frames based on Timoshenko beam theory was presented by Yu et al.[1] using the exact transfer dynamic stiffness matrix. The inverse of the block matrix $\boldsymbol{T}_{df}$ of the transfer matrix may encounter numerical instability when computed with a fixed floating point.

$$|\boldsymbol{f}_{xy}| = g\begin{vmatrix} \lambda_1(\sigma-\lambda_2^2)\sinh(\lambda_1 x/l)+\lambda_2(\sigma+\lambda_1^2)\sin(\lambda_1 x/l) & \cosh(\lambda_1 x/l)-\cos(\lambda_2 x/l) \\ (\sigma+\lambda_1^2)(\sigma-\lambda_2^2)[\cosh(\lambda_1 x/l)-\cos(\lambda_2 x/l)] & \left[(\sigma+\lambda_1^2)/\lambda_1\right]\sinh(\lambda_1 x/l)+\left[(\sigma-\lambda_2^2)/\lambda_2\right]\sin(\lambda_2 x/l)\end{vmatrix}$$
$$= g\Big(\left[\lambda_1(\sigma-\lambda_2^2)\sinh(\lambda_1 x/l)+\lambda_2(\sigma+\lambda_1^2)\sin(\lambda_2 x/l)\right]\left\{\left[(\sigma+\lambda_1^2)/\lambda_1\right]\sinh(\lambda_1 x/l)+\left[(\sigma-\lambda_2^2)/\lambda_2\right]\sin(\lambda_2 x/l)\right\}$$
$$-(\sigma+\lambda_1^2)(\sigma-\lambda_2^2)[\cosh(\lambda_1 x/l)-\cos(\lambda_2 x/l)]^2\Big) \tag{5}$$

The transfer matrix relationship in Ref. 1 was given as

$$\boldsymbol{S}_x = \boldsymbol{T}(x)\cdot\boldsymbol{S}_0 = \boldsymbol{B}(x)\boldsymbol{B}(0)^{-1}\cdot\boldsymbol{S}_0 \tag{1}$$

which was partitioned in the form

$$\boldsymbol{S}_x = \begin{Bmatrix} d_x \\ f_x \end{Bmatrix} = \begin{bmatrix} \boldsymbol{T}_{dd} & \boldsymbol{T}_{df} \\ \boldsymbol{T}_{fd} & \boldsymbol{T}_{ff} \end{bmatrix}\begin{Bmatrix} d_0 \\ f_0 \end{Bmatrix} = \boldsymbol{T}(x)\cdot\boldsymbol{S}_0 \tag{2}$$

In Eqs. (1) and (2), $\boldsymbol{S}$, $\boldsymbol{d}$ and $\boldsymbol{f}$ are the state vector, displacement vector, and force vector, respectively, the subscripts x and 0 denote an arbitrary point x and the beam end $x=0$, and $\boldsymbol{T}(x)$ is the transfer matrix. Equation (2) was transferred to the force–displacement relationship through a stiffness matrix as follows:

$$\begin{Bmatrix} f_0 \\ f_x \end{Bmatrix} = \begin{bmatrix} -\boldsymbol{T}_{df}^{-1}\boldsymbol{T}_{dd} & \boldsymbol{T}_{df}^{-1} \\ \boldsymbol{T}_{fd}-\boldsymbol{T}_{ff}\boldsymbol{T}_{df}^{-1}\boldsymbol{T}_{dd} & \boldsymbol{T}_{ff}\boldsymbol{T}_{df}^{-1} \end{bmatrix}\begin{Bmatrix} d_0 \\ d_x \end{Bmatrix} \tag{3}$$

The matrix $\boldsymbol{T}_{df}$ is the upper right block matrix of dimension 6×6 whose explicit form can be calculated as

$$\boldsymbol{T}_{df} = \begin{bmatrix} \boldsymbol{f}_{xy} & & & \\ & \boldsymbol{f}_{xz} & & \\ & & a & \\ & & & t \end{bmatrix}_{6\times 6} \tag{4}$$

where $\boldsymbol{f}_{xy}$ and $\boldsymbol{f}_{xz}$ are the 2×2 matrices relating to the flexural vibrations in $x-y$ and $x-z$ planes, respectively, whereas a and t are two scalars relating to the axial and torsional vibrations respectively. The entries not shown in the matrix $\boldsymbol{T}_{df}$ are zero.

Note that the reversibility of a square matrix necessitates a nonzero determinant of the matrix. Hence, it is essential to investigate the determinant $\det(\boldsymbol{T}_{df}) = at|\boldsymbol{f}_{xy}\|\boldsymbol{f}_{xz}|$. For illustration, we only consider the determinant of the block matrix $\boldsymbol{f}_{xy}$, whose explicit form is

where $g = l^4/[(EI_z)^2\beta^4(\lambda_1^2+\lambda_2^2)^2]$. Expanding the square brackets yields

$$|\boldsymbol{f}_{xy}| = g\Big\{\left[\lambda_2(\sigma+\lambda_1^2)^2/\lambda_1 - \lambda_1(\sigma-\lambda_2^2)^2/\lambda_2\right]\sinh(\lambda_1 x/l) \times \sin(\lambda_2 x/l) - 2(\sigma+\lambda_1^2)(\sigma-\lambda_2^2) \times [1+\cosh(\lambda_1 x/l)\cos(\lambda_2 x/l)]\Big\} \tag{6}$$

We see that all of the hyperbolic and trigonometric functions are responsible for the results of the determinant. However, numerical instability arises when performing the calculation numerically with a fixed floating point. In fact, when we note that the elements in the determinant are numbers, the influence of $\sin(\lambda_2 x/l)$ and $\cos(\lambda_2 x/l)$ will be swamped by $\sinh(\lambda_1 x/l)$ and $\cosh(\lambda_1 x/l)$ respectively, when the real part of the eigenvalue λ_1, $\mathrm{Re}(\lambda_1)$, is a very large positive number. This numerical difficulty has been discussed previously by many researchers in seismology as well as structural engineering. In this situation, the determinant of $\boldsymbol{f}_{xy}$, when calculated numerically based on Eq. (5), becomes

$$|\boldsymbol{f}_{xy}| = g(\sigma+\lambda_1^2)(\sigma-\lambda_2^2)\left[\sinh^2(\lambda_1 x/l)-\cosh^2(\lambda_1 x/l)\right] \tag{7}$$

When the trigonometric terms are swamped as mentioned, the difference between $\sinh^2(\lambda_1 x/l)$ and $\cosh^2(\lambda_1 x/l)$ becomes indiscriminable in the numerical calculation, and hence, the right-hand side of Eq. (7) vanishes. Thus, if the inversion of $\boldsymbol{T}_{df}$ is performed numerically, then numerical instability arises when eigenvalues with large real part are involved.

Table 1 presents an illustration of the numerical computation of the hyperbolic functions. It is seen that, as soon as the argument ξ is large enough, the number 1 is swamped by the hyperbolic

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Table 1 Numerical calculation of hyperbolic functions employing software with certain floating point calculation ability

Function	Exact form	ξ	Numerical value	Exact value
$[\sinh(\xi)+1]^2-[\sinh(\xi)-1]^2$	$4\sinh(\xi)$	38.07	0	6.83×10^{16}
$\sinh^2(\xi)-\cosh^2(\xi)$	-1	18.72	0	-1

functions and, of course, the trigonometric functions (always smaller than 1) will be swamped. Therefore, entirely wrong numerical results will be obtained. Table 1 also shows that the argument ξ of $\sinh^2(\xi)-\cosh^2(\xi)$ for numerical swamping is far lower than that of $[\sinh(\xi)+1]^2-[\sinh(\xi)-1]^2$.

We hope, through this Comment, that the reader who is not familiar with the history of structural matrix method widely used in 1950–1960s will become aware of this numerical instability. Several effective, although not universal, methods have been proposed to overcome the difficulty, as summarized by Pestel and Leckie.[2]

References

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[2] Pestel, E. C., and Leckie, F. A., *Matrix Methods in Elastomechanics*, McGraw–Hill, New York, 1963, pp. 192–213.

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